HEAT TRANSFER IN AN UNEVENLY COOLED ROD

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The method of [1, 2] is used to consider the heating of a thin semiinfinite rod from the end, this being unevenly cooled on the sides; the basis of the method is presented.

Consider the problem

$$\left[\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}+\gamma(x)\right]T(x,\ t)=0;\quad x\geqslant 0,\quad t>0; \tag{1}$$

$$T|_{x=0}=T_0(t); \tag{2}$$

$$T|_{x=\infty}=0; \tag{3}$$

$$T|_{t=0}=0. \tag{4}$$

We have to find the temperature gradient at the end of the rod $q_0 = (\partial T/\partial x)_{x=0}$ (this quantity at once determines the heat flux).

This formulation (which does not require the temperature distribution to be derived) is very frequently encountered in applied studies, but to solve it by traditional techniques one has to find first of all the value of T(x, t) for each instant and each point in the rod. The proposed method enables one to determine $q_0(t)$ without solving (1)-(4) completely. We represent (1) as a product of two operators:

$$\left[\sum_{m=0}^{\infty} b_m(x)\frac{\partial^{\frac{1-m}{2}}}{\partial t^{\frac{1-m}{2}}}-\frac{\partial}{\partial x}\right]\left[\sum_{n=0}^{\infty} a_n(x)\frac{\partial^{\frac{1-n}{2}}}{\partial t^{\frac{1-n}{2}}}+\frac{\partial}{\partial x}\right]T=0, \tag{5}$$

each of which is dependent only on the first derivative with respect to x; here b_m and a_n are unknown functions. The operators for the fractional derivatives are defined as follows [3]:

$$\frac{d^\nu f(t)}{dt^\nu}=\frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{-\nu}d\tau;\quad -\infty<\nu<1. \tag{6}$$

For $\nu < 0$ we integrate (6) by parts to get another form for the definition:

$$\frac{d^\nu f(t)}{dt^\nu}=\frac{1}{\Gamma(-\nu)}\int_0^t f(\tau)(t-\tau)^{-\nu-1}d\tau. \tag{6a}$$

The following relationships apply for the fractional derivatives of (6):

$$\frac{d^\nu}{dt^\nu}\frac{d^\mu}{dt^\mu}f(t)=\frac{d^{\nu+\mu}}{dt^{\nu+\mu}}f(t),\quad \nu,\ \mu<1,\quad \nu+\mu\leqslant 1; \tag{7}$$

$$\frac{d^\nu}{dt^\nu}f(t)\,g(t)=\sum_{n=0}^{\infty}\binom{\nu}{h}f^{(n)}_{(t)}\frac{d^{\nu-n}g(t)}{dt^{\nu-n}}; \tag{8}$$

$$\frac{d^\nu t^\mu}{dt^\nu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)}t^{\mu-\nu},\quad \mu>-1; \tag{9}$$

$$\overline{\frac{d^\nu f(t)}{dt^\nu}}=p^\nu\bar{f}(p) \tag{10}$$

(a bar denotes the Laplace transformation with respect to t).

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Fractional differentiation is familiar in analysis; the integrals of (6a) have been considered in detail in many works [4, 5]. A detailed survey of the history of fractional differentiation in the nineteenth century is to be found in [8]. The operation is frequently used in inexplicit form, as in transferring in Laplace-transform space if one uses the operator p^{ν} (ν fractional).

The language of fractional differentiation is most convenient for presenting the present method, which consists as follows. We multiply the operators in (5) and use the property of (7) to equate the expressions for identical powers of the derivatives to the corresponding expressions in (1), which gives us a system of recurrence relationships for a_n and b_n. We find that $a_n = b_n$, $a_0 = 1$, $a_1 = 0$ and

$$\begin{aligned} 2a_2 &= \gamma(x), \\ 2a_3 &= a_2', \\ 2a_4 &= a_3' - a_2^2, \\ 2a_5 &= a_4' - a_2 a_3 - a_3 a_2, \\ &\ldots\ldots\ldots\ldots \\ 2a_n &= a_{n-1}' - \sum_{k=0}^{n-4} a_{k+2} a_{n-k-2}, \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{11}$$

We now consider the equation formed by the right factor of the operator in (5):

$$\left[\sum_{n=0}^{\infty} a_n(x) \frac{\partial^{\frac{1-n}{2}}}{\partial t^{\frac{1-n}{2}}} + \frac{\partial}{\partial x}\right] T = 0. \tag{12}$$

It will be shown below that this equation has a solution satisfying (2)-(4) subject to certain reservations; this is the basic content of the method. The equation is of first order in x, and the right factor has solutions that automatically satisfy the condition of being bounded at infinity, and so one can consider (12) in what follows in place of (1). This is very much more convenient for the present problem, since it enables one to determine the flux at the boundary of the region directly.

If the series in (12) converges uniformly in t, we apply a Laplace transformation with respect to t to (12) and use (10) to get

$$\left[\sum_{n=0}^{\infty} a_n(x) p^{\frac{1-n}{2}} + \frac{d}{dx}\right] \bar{T} = 0.$$

The latter can be integrated, and condition (2) is met:

$$\bar{T} = \bar{T}_0(p) \exp\left[-\int_0^x \sum_{n=0}^{\infty} a_n(x) p^{\frac{1-n}{2}} dx\right]. \tag{13}$$

If we have

$$\lim_{n\to\infty} \left| \frac{a_n(x) \sqrt{p}}{a_{n+1}(x)} \right| > 1 \tag{14}$$

for all x for $p > p^*$, then the series in (13) converges absolutely and uniformly for all $p > p^*$ and represents a solution to (12) in operator form. Then there exists an original for $\bar{T}(x, p)$ for all $0 < t < \infty$: the solution to (13) satisfies also (3) if $\lim_{x\to\infty} \sum_{n=0}^{\infty} a_n(x) p^{(1-n)/2} > 0$ ($\mathrm{Re}\, p > 0$), which is true, for instance, for an analytical $\gamma(x)$ such that $\lim_{x\to\infty} \gamma(x) = \mathrm{const} \geq 0$; condition (4) is also satisfied if $T_0(0) = 0$, since

$$\lim_{t\to+0} T(x, t) = \lim_{p\to\infty} \bar{T}(x, p) = \lim_{p\to\infty} p\bar{T}_0(p) \exp[-\sqrt{p}x - O(1/\sqrt{p})]$$

for all x (checks on particular examples show that the method gives a correct result also for $T_0(0) = \mathrm{const}$). As the solution to (12) is also a solution to (1) under the above conditions, then (13) will be simultaneously a solution to (1)-(4) in operator form.

If (14) is met, the series in (12) converges absolutely and uniformly for all x and t; the definition of (6a) indicates that for $\nu < 0$ one can use the theorem on the mean to obtain an estimate for the fractional derivative of the bounded function

$$\left|\frac{\partial^{\nu} T}{\partial t^{\nu}}\right| \leqslant \sup |T| \frac{t^{-\nu}}{\Gamma(1-\nu)} . \tag{15}$$

If $T|_{t=0} = 0$, we integrate (6) by parts and apply the theorem on the mean to get

$$\left|\frac{\partial^{1/2} T}{\partial t^{1/2}}\right| \leqslant \sup \left|\frac{\partial T}{\partial t}\right| \frac{\sqrt{t}}{\Gamma(3/2)} . \tag{15a}$$

As T and $\partial T/\partial t$ are bounded for a bounded $T_0(t)$, as (13) shows, the series in (12) converges better than the absolutely and uniformly convergent series

$$\sup\left|\frac{\partial T}{\partial t}\right| \frac{\sqrt{t}}{\Gamma(3/2)} + \sup|T| \sum_{n=1}^{\infty} (p^*)^{-n/2} \frac{t^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} .$$

It follows from the estimates of (15) that derivatives of order $\nu \leq 1/2$ are continuous in t for $t \to +0$ for $T_0(0) = 0$; also, the quantities $a_n(x)$ are continuous for $x = +0$ for the analytical function γ, so we consider the limit of (12) for $x \to +0$ to write down at once the solution to the problem in the form

$$-q_0(t) = \sum_{n=0}^{\infty} a_n(0) \frac{d^{\frac{1-n}{2}} T_0(t)}{dt^{\frac{1-n}{2}}} . \tag{16}$$

Then the derivation of $q_0(t)$ for an arbitrary $\gamma(x)$ amounts to determining $a_n(0)$ from (11); we enumerate again the conditions under which the proof has been derived: 1) $T_0(t)$ is a bounded differentiable function such that $T_0(0) = 0$; 2) $\gamma(x) \geq 0$ is analytical and such that $\gamma(\infty) = \text{const}$. Also, $a_n(x)$ must meet (14).

We do not know the general conditions for (14) to be met, but we have obtained a sufficient condition. Consider the accessory operator L specified by the relationship

$$L^2 f(x) = \left[\frac{d}{dx} + \gamma(2x)\right] f(x). \tag{17}$$

We seek L as a series in fractional derivatives:

$$Lf(x) = \sum_{n=0}^{\infty} h_n(x) \frac{d^{\frac{1}{2}-n}}{dx^{\frac{1}{2}-n}} f(x). \tag{18}$$

We substitute (18) into (17) and use (7) and (8) to get a system of recurrence relationships for $h_n(x)$:

$$\begin{gathered}
h_0 = 1, \\
2h_1 = \gamma(2x), \\
2h_2 = -h_1^2 - \binom{\frac{1}{2}}{1} h_1', \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
2h_n = -\sum_{k=1}^{n-1} h_k h_{n-k} - \frac{1}{2} h_{n-1}' - \sum_{s=1}\sum_{k=1}^{k+s\leq n} \binom{\frac{1}{2}-k}{s} h_k h_{n-k-s}^{(s)}. \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{gathered} \tag{19}$$

If $\gamma(x) \geq 0$ and all $\gamma^{(n)}(x) \geq 0$, the signs of all terms on the right in each equation are identical, while the signs of the $h_n(x)$ alternate; it is clear that the $|h_n|$ exceed the terms $|c_n|$ of the system

$$\begin{gathered}
2c_1 = \gamma(x), \\
2c_2 = c_1', \\
2c_3 = c_2' + c_2 c_1 + c_1 c_2, \\
\cdots\cdots\cdots\cdots \\
2c_n = c_{n-1}' + \sum_{k=1}^{n-1} c_k c_{n-k-1}, \\
\cdots\cdots\cdots\cdots
\end{gathered} \tag{20}$$

which in turn exceeds system (11), as $|c_n| \geq |a_{n-1}|$.

On the other hand, we can find the exact values of the $h_n(x)$ on the basis that

$$Lf(x) = e^{-\int_0^x \gamma(2x)dx} \frac{d^{1/2}}{dx^{1/2}} e^{\int_0^x \gamma(2x)dx} f(x), \tag{21}$$

as can be seen by direct test on (17); we expand (21) as the series of (18) with the condition of (8) to get

$$h_n(x) = \binom{\frac{1}{2}}{n} e^{-\int_0^x \gamma(2x)dx} \frac{d^n}{dx^n} e^{\int_0^x \gamma(2x)dx}. \tag{22}$$

As $|a_{n-1}| \leq |c_n| \leq |h_n|$ and as $\lim\limits_{n\to\infty} \binom{\frac{1}{2}}{n} \sim O(1/\sqrt{n})$, we get the sufficient condition for (14) to be met:

$$\lim_{n\to\infty} e^{-\int_0^x \gamma(2x)dx} \frac{d^n}{dx^n} e^{\int_0^x \gamma(2x)dx} \leqslant O(1/\sqrt{n}). \tag{23}$$

If any of the derivatives $\gamma^{(n)}$ are negative, the relevant h_n should be determined from (23), with γ replaced by γ^*, which is derived from γ by replacing all the negative derivatives by positive ones.

The following are the expressions for some of the first $a_n(x)$:

$$a_0 = 1; \quad a_1 = 0; \quad a_2 = \frac{\gamma}{2}; \quad a_3 = \frac{\gamma'}{4}; \quad a_4 = \frac{\gamma''}{8} - \frac{\gamma^2}{8};$$

$$a_5 = \frac{\gamma'''}{16} - \frac{\gamma\gamma'}{4}; \quad a_6 = \frac{\gamma^{IV}}{32} - \frac{5}{32}\gamma'^2 - \frac{3}{16}\gamma\gamma'' + \frac{\gamma^3}{16};$$

$$a_7 = \frac{1}{64}\gamma^{V} - \frac{9}{32}\gamma''\gamma' - \frac{1}{8}\gamma\gamma''' + \frac{1}{4}\gamma^2\gamma';$$

$$a_8 = \frac{1}{128}\gamma^{VI} - \frac{19}{128}\gamma''^2 - \frac{7}{32}\gamma'\gamma''' - \frac{5}{64}\gamma\gamma^{IV} + \frac{25}{64}\gamma\gamma'^2 + \frac{15}{64}\gamma^2\gamma'' - \frac{5}{128}\gamma^4;$$

$$a_9 = \frac{1}{256}\gamma^{VII} - \frac{17}{64}\gamma''\gamma''' - \frac{5}{32}\gamma'\gamma^{IV} - \frac{3}{64}\gamma\gamma^{V} + \frac{15}{64}\gamma'^3 + \frac{27}{32}\gamma\gamma'\gamma'' + \frac{3}{16}\gamma^2\gamma''' - \frac{1}{4}\gamma^3\gamma'; \tag{24}$$

$$a_{10} = \frac{1}{512}\gamma^{VIII} - \frac{69}{512}\gamma'''^2 - \frac{55}{256}\gamma^{IV}\gamma'' - \frac{27}{256}\gamma'\gamma^{V} - \frac{7}{256}\gamma\gamma^{VI}$$
$$+ \frac{221}{256}\gamma'^2\gamma'' + \frac{133}{256}\gamma\gamma''^2 + \frac{49}{64}\gamma\gamma'\gamma''' + \frac{35}{256}\gamma^2\gamma^{IV} - \frac{175}{256}\gamma^2\gamma'^2 - \frac{35}{128}\gamma^3\gamma'' + \frac{7}{256}\gamma^5;$$

$$a_{11} = \frac{1}{1024}\gamma^{IX} - \frac{125}{512}\gamma'''\gamma^{IV} - \frac{83}{512}\gamma''\gamma^{V} - \frac{35}{512}\gamma'\gamma^{VI} - \frac{1}{64}\gamma\gamma^{VII}$$
$$+ \frac{153}{128}\gamma'\gamma''^2 - \frac{225}{256}\gamma'^2\gamma''' + \frac{17}{16}\gamma\gamma''\gamma''' + \frac{5}{8}\gamma\gamma'\gamma^{IV} + \frac{3}{32}\gamma^2\gamma^{V} - \frac{15}{16}\gamma\gamma'^3 - \frac{27}{16}\gamma^2\gamma'\gamma'' - \frac{1}{4}\gamma^3\gamma''' + \frac{1}{4}\gamma^4\gamma'.$$

We now discuss the advantages and disadvantages of this method relative to the classical method of finding q_0 from (1)-(4), which is as follows. Equation (1) after Laplace transformation becomes

$$\left\{ \frac{d^2}{dx^2} - [p + \gamma(x)] \right\} \overline{T} = 0. \tag{25}$$

We find the solution to the latter that satisfies the conditions of (2)-(4), which is differentiated and written for $x = 0$; after transfer to the original we get the functional relationship between q_0 and T_0.

The advantages of this method are as follows: 1. We do not need to solve (25) with variable coefficients; the operations involved in finding the a_n appearing in (16) are very simple, since they involve only differentiation and algebraic transformation. In passing we have found a general solution to (25) in the form of (13) that satisfies the condition of being bounded for $x \to \infty$. 2. The method is applicable without substantial change to problems with inseparable variables [1, 2], which cannot be solved by the classical method.

Deficiencies of the method: 1. The solution is obtained as a series of special form convenient for practical calculations for not very large t. To simplify the solution for a wide range in t one has either to sum the series in (16) or else to substitute manually a sufficiently large number of a_n from (11). Then the classical method is more applicable if the exact solution to (25) is known. 2. The solution can be obtained by this method in a very simple fashion only for a semiinfinite region; if the region is finite, one has to consider solutions given by both factors in (5). It is found that then it is impossible to derive exact values for the a_n and b_m in (5) for arbitrary γ. On the other hand, the solution by classical methods for a finite region is in no way more complicated than that for an infinite one. For instance, [6, 7] give a solution for a finite interval, from which one can easily find the relationship between q_0 and T_0 (in our symbols).

Example. Let $T_0(t) = \alpha t$, $\gamma(x) = \beta x$; (23) is met, and then from (16) with (9) and (11) we get

$$-\alpha^{-1} q_0(t) = \frac{t^{1/2}}{\Gamma(3/2)} + \frac{\beta}{4}\frac{t^2}{\Gamma(3)} - \frac{5\beta^2}{32}\frac{t^{7/2}}{\Gamma(9/2)} + \frac{15\beta^3}{64}\frac{t^5}{\Gamma(6)} - \frac{295\beta^4}{512}\frac{t^{13/2}}{\Gamma(15/2)} + \dots \quad (26)$$

For a steel rod ($\varkappa = 2\cdot 10^{-5}$ m^2/sec, $R = 10^{-2}$ m) with air blown over it at room temperature we assume $\beta = 2\cdot 10^2$ 1/m^3, and the last term written out in (26) is 1% of the first up to $t \approx 1200$ sec.

This method is applicable also to calculating periodic processes, where instead of (6) we use a different definition for fractional differentiation. For the periodic function $\cos\omega t$ we have

$$\frac{d^\nu \cos\omega t}{dt^\nu} = \omega^\nu \cos\left(\omega t + \frac{\pi}{2}\nu\right), \quad (27)$$

and the solution for a periodic $T_0(t)$ is given by (16), where the expression of (27) is to be understood for the symbols for fractional differentiation.

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NOTATION

x	is the coordinate;
t	is the time;
T	is the temperature;
T_0	is the temperature at rod end;
q_0	is the temperature gradient at rod end;
a, b, c, h	are the coordinate functions in the fractional derivative series;
L	is the auxiliary operator;
R	is the rod radius;
p	is the variable in Laplace transform variable;
α, β	are constants in the numerical example;
γ	is the variable heat-transfer coefficient;
$\varkappa$	is the thermal diffusivity;
ω	is the frequency;
m, n, k, s, μ, ν	are the summation and differentiation indices.

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